Three-slit experiments and quantum nonlocality - The absence of 3rd-order interference implies Tsirelson's bound

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Abstract. An interesting link between two very different physical aspects of quantum mechanics is revealed; these are the absence of third-order interference and Tsirelson's bound for the nonlocal correlations. Considering multiple-slit experiments - not only the traditional configuration with two slits, but also configurations with three and more slits - Sorkin detected that third-order (and higher-order) interference is not possible in quantum mechanics. The EPR experiments show that quantum mechanics involves nonlocal correlations which are demonstrated in a violation of the Bell or CHSH inequality, but are still limited by a bound discovered by Tsirelson. It now turns out that Tsirelson's bound holds in almost any other probabilistic theory provided that a reasonable calculus of conditional probability is included and third-order interference is ruled out.

Key words. Quantum theory; nonlocality; Tsirelson's bound; three-slit experiment; higher-order interference; conditional probability; quantum logic

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1 Introduction

The EPR experiments show that quantum mechanics involves nonlocal correlations which are demonstrated in a violation of the Bell inequalities [1] or CHSH inequality [2] and have nowadays become the core of quantum information theory.

However, the nonlocal quantum correlations are not unlimited; they obey a bound found by Tsirelson [13]. It follows algebraically from Hilbert space quantum mechanics without revealing any deeper physical or information theoretical principle behind it. The search for such a principle is a matter of ongoing research. Janotta et al. consider a special geometric property of the state space, known as strong self-duality [4]. Pawlowski et al. have recently introduced a promising new principle called information causality [9].

The present paper establishes a link between Tsirelson's bound for the quantum correlations and another interesting property of quantum mechanics which was discovered by Sorkin [12]. Considering multiple-slit experiments - not only the traditional configuration with two slits, but also configurations with three and more slits - Sorkin detected that quantum mechanics rules out third-order interference.

It will be shown in the following sections that the absence of third-order interference implies the validity of Tsirelson's bound. The correlations in any probabilistic theory cannot exceed Tsirelson's bound if a reasonable calculus of conditional probability is included and third-order interference is excluded. Mathematically, this is a quite simple consequence of the results from Ref. [8], but nevertheless provides an interesting link between two very different physical aspects of quantum mechanics.

The general framework for a probabilistic theory from Refs. [5], [6] and [8] and Sorkin's concept of third-order interference [12] are briefly depicted in sections 2 and 3 before then turning to the major results in sections 4, 5 and 6. A general mathematical inequality which follows from the absence of third-order interference is presented in section 4. Its relation to Tsirelson's bound is pointed out in section 5. This requires an understanding of the meaning of observables and compatibility in the general framework, which is elaborated in section 6.

2 The probabilistic framework

In quantum mechanics, the measurable quantities of a physical system are represented by observables. Most simple are those observables where only the two discrete values 0 and 1 are possible as measurement outcome; they are called *events* or *propositions* and are elements of a mathematical structure called *quantum logic*.

Standard quantum mechanics uses a very special type of quantum logic; it consists of the self-adjoint projections on a Hilbert space or, more generally, of the self-adjoint projections in a von Neumann algebra.

An abstractly defined quantum logic contains two specific elements 0 and $\mathbb{I}$ and possesses an *orthogonality relation*, an *orthocomplementation* and a *partial sum operation* + which is defined only for orthogonal events [5], [8].

The *states* on a quantum logic are the analogue of the probability measures in classical probability theory, and *conditional probabilities* can be defined similar to their classical prototype [5], [8]. A state μ allocates the probability $\mu(f)$ to each event f (element of the quantum logic), and the conditional probability of an event f under another event e is the updated probability after the outcome of a first measurement has been the event e; it is denoted by $\mu(f \mid e)$.

However, among the abstractly defined quantum logics, there are many where no states or no conditional probabilities exist, or where the conditional probabilities are ambiguous. Therefore, only those quantum logics where sufficiently many states and unique conditional probabilities exist can be considered a satisfying framework for general probabilistic theories. Such a quantum logic generates an order-unit space A and can be embedded in the unit interval $[0, \mathbb{I}] := \{a \in A : 0 \leq a \leq \mathbb{I}\}$ of this order-unit space; $\mathbb{I}$ becomes the order-unit [8]. The elements of A are candidates for the observables.

Not much knowledge of order-unit spaces is required for the understanding of the present paper and the interested reader is referred to the monograph [3].

3 Third-order interference

Consider the following mathematical term I_3 for a triple of orthogonal events e_1, e_2 and e_3, a further event f and a state μ:

$$\begin{aligned} I_3 \quad := \quad & \mu(f \mid e_1 + e_2 + e_3)\,\mu(e_1 + e_2 + e_3) \\ & -\mu(f \mid e_1 + e_2)\,\mu(e_1 + e_2) \\ & -\mu(f \mid e_1 + e_3)\,\mu(e_1 + e_3) \\ & -\mu(f \mid e_2 + e_3)\,\mu(e_2 + e_3) \\ & +\mu(f \mid e_1)\,\mu(e_1) + \mu(f \mid e_2)\,\mu(e_2) + \mu(f \mid e_3)\,\mu(e_3) \end{aligned}$$

This term I_3 was introduced by Sorkin [12], and he detected that $I_3 = 0$ is universally valid in quantum mechanics. His original definition refers to probability measures on 'sets of histories'. With the use of conditional probabilities, it gets the above shape, which was seen by Ududec, Barnum and Emerson [14].

For the three-slit set-up which Sorkin considered, the identity $I_3 = 0$ means that the interference pattern observed with three open slits $(e_1 + e_2 + e_3)$ is a simple combination of the patterns observed in the six different cases when only one or two among the three available slits are open $(e_1, e_2, e_3, e_1 + e_2, e_1 + e_3$, and $e_2 + e_3)$. This could be confirmed in a recent experimental test with photons [11].

The new type of interference which is present whenever $I_3 \neq 0$ holds is called *third-order interference.*

4 A mathematical inequality

Quantum logics which do not exhibit third-order interference (i.e., which satisfy the identity $I_3 = 0$) have been studied in Ref. [8], and it has been shown that there is a product operation $\Box$ in the order-unit space A generated by the quantum logic, if the Hahn-Jordan decomposition property holds in addition ([8] Lemma 10.2). The Hahn-Jordan decomposition for quantum logics is the analogue of the Hahn-Jordan decomposition for signed measures in the classical case; it is a mathematical technical requirement the details of which are beyond the scope of the present paper and can be found in Ref. [8].

The product $a \Box b$ is linear in a as well as in b and satisfies the inequality $\|a \Box b\| \leq \|a\| \; \|b\|$ $(a, b \in A)$, where $\| \; \|$ denotes the order-unit norm on A. The events e become idempotent elements in A (i.e., $e = e^2 = e \Box e$), and $e \Box f = 0$ for any orthogonal event pair e and f. Generally, however, the product is neither commutative nor associative nor power-associative. Moreover, the square $a^2 = a \Box a$ of an element a in A need not be positive.

If $a^2 \geq 0$ holds for all $a \in A$, then any $a_1, a_2, b_1, b_2 \in [-\mathbb{I}, \mathbb{I}] := \{a \in A : -\mathbb{I} \leq a \leq \mathbb{I}\}$ $= \{a \in A : \|a\| \leq 1\}$ satisfy the following inequality:

$$\begin{aligned} -4\sqrt{2}\mathbb{I} \quad &\leq \quad a_1 \Box b_1 + b_1 \Box a_1 + a_1 \Box b_2 + b_2 \Box a_1 \\ &\quad\quad +a_2 \Box b_1 + b_1 \Box a_2 - a_2 \Box b_2 - b_2 \Box a_2 \qquad (1) \\ &\leq \quad 4\sqrt{2}\,\mathbb{I} \end{aligned}$$

Proof. $0 \leq ((1 + \sqrt{2})(a_1 - b_1) + a_2 - b_2)^2 + ((1 + \sqrt{2})(a_1 - b_2) - a_2 - b_1)^2$

$$+((1 + \sqrt{2})(a_2 - b_1) + a_1 + b_2)^2 + ((1 + \sqrt{2})(a_2 + b_2) - a_1 - b_1)^2 =$$

$4(2 + \sqrt{2})(a_1^2 + a_2^2 + b_1^2 + b_2^2)$

$-4(1+\sqrt{2})(a_1 \Box b_1 + b_1 \Box a_1 + a_1 \Box b_2 + b_2 \Box a_1 + a_2 \Box b_1 + b_1 \Box a_2 - a_2 \Box b_2 - b_2 \Box a_2))$

The second $\leq$-sign in inequality (1) now follows from $a_1^2 + a_2^2 + b_1^2 + b_2^2 \leq 4\,\mathbb{I}$, the first one follows by exchanging a_1 with $-a_1$ and a_2 with $-a_2$.

This is a simple transfer of Tsirelson's proof [13] from quantum mechanics to the more general setting. However, this becomes possible only by using a deep result (Lemma 10.2) from Ref. [8]. The meaning of the above inequality and its relation to Tsirelson's bound for quantum mechanical correlations shall be studied in the following two sections.

5 Tsirelson's bound

The EPR experiments exhibit that quantum mechanics involves stronger correlations between two spatially separated physical systems than possible in the classical case. Suppose that a_1 and a_2 are observables of the first system and b_1 and b_2 observables of the second system and that the spectrum of each observable lies in the interval $[-1, 1]$. Usually, it is assumed that a_k and b_l are compatible, i.e., a_k commutes with b_l $(k, l = 1, 2)$, but neither a_1 and a_2 nor b_1 and b_2 need commute. The expectation values c_{kl} of the products $a_k b_l$ $(k, l = 1, 2)$ are a measure for the correlations between the two systems then.

If all four observables would commute with each other or if they were classical random variables, it would follow that

$$\begin{aligned} |a_1b_1 + a_1b_2 + a_2b_1 - a_2b_2| &\leq |a_1|\ |b_1 + b_2| + |a_2|\ |b_1 - b_2| \\ &\leq |b_1 + b_2| + |b_1 - b_2| \\ &\leq 2 \end{aligned}$$

(note that commuting observables can be considered as functions) and therefore

$$|c_{11} + c_{12} + c_{21} - c_{22}| \leq 2$$

This is the CHSH inequality [2]. However, if neither a_1 and a_2 nor b_1 and b_2 commute, $c_{11} + c_{12} + c_{21} - c_{22}$ can exceed the value 2 and can reach the value $2\sqrt{2}$ in certain EPR experiments. This shows that quantum correlations do not obey the same rules as classical correlations.

Tsirelson [13] discovered that $2\sqrt{2}$ is the largest possible value for $c_{11} + c_{12} + c_{21} - c_{22}$ in quantum mechanics. Inequality (1) now shows that Tsirelson's bound still holds in a more general theory. It remains valid if the theory includes a reasonable calculus of conditional probability, rules out third-order interference, satisfies the Hahn-Jordan property and if squares in A are positive. However, this requires an understanding of the meaning of observables and compatibility in the general framework, which shall be discussed in the following section.

6 Observables and compatibility

Suppose that third-order interference is ruled out, that the Hahn-Jordan property holds and that squares are positive in the order-unit space A. Is each element in A an observable then? An observable represents a measurable physical quantity and should lie in an associative commutative algebra, ideally in an algebra of real functions.

It can indeed be shown that any associative commutative closed subalgebra of A containing the order-unit $\mathbb{I}$ is an associative JB-algebra (JB-algebras are the Jordan analogue of the C^* -algebras; see [3]) and thus isomorphic to the algebra of continuous real functions on some compact Hausdorff space; this can be proved in the same way as a very similar result in Ref. [6]. Therefore, only those elements of A which generate an associative subalgebra represent observables.

Simple examples are the elements having the shape $\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$ with mutually orthogonal events $e_1, e_2, ..., e_n$ and real numbers $\alpha_1, \alpha_2, ..., \alpha_n$. Note that $e_k \Box e_l = 0$ for $k \neq l$ because of the orthogonality and that $e_k \Box e_k = e_k$.

Two observables are *compatible* if they lie in a joint associative commutative subalgebra of A. The pair then behaves like two classical random variables and they are simultaneously measurable. This represents the strongest level in the hierarchy of different compatibility and comeasurability levels studied in Ref. [7]. There, this strong level was named '*algebraic compatibility*.' Since the distinction among the different levels is not needed in the present paper, the tag 'algebraic' is dropped here.

A state becomes a positive linear functional ρ on the order-unit space A satisfying $\rho(\mathbb{I}) = 1$, and $\rho(a)$ is the expectation value of an observable a in A. For a compatible observable pair a and b, $a \Box b = b \Box a$ holds, the product itself is an observable, and $\rho(a \Box b)$ is a measure for their correlation.

With four observables $a_1, a_2, b_1, b_2 \in [-\mathbb{I}, \mathbb{I}]$ such that a_k and b_l are compatible $(k, l = 1, 2)$, inequality (1) yields that

$$|c_{11} + c_{12} + c_{21} - c_{22}| \leq 2\sqrt{2}$$

holds for the correlations $c_{kl} = \rho(a_k \Box b_l) = \rho(b_l \Box a_k)$ $(k, l = 1, 2)$. This means that Tsirelson's bound remains valid in this setting, although it is more general than quantum mechanics.

7 Conclusion

Using mathematical methods, an interesting link between two very different physical aspects of quantum mechanics has been revealed; it has been seen that Tsirelson's bound holds for the correlations in any probabilistic theory if third-order interference is ruled out. Three further assumptions are needed: a reasonable calculus of conditional probability, the Hahn-Jordan decomposition property, which is a technical mathematical requirement, and the positivity of squares of the elements in the order-unit space A.

If it is assumed that each element in A is an observable, A becomes a so-called JBW-algebra (the Jordan algebra analogue of a W^*-algebra or von Neumann algebra; see [3]). This is the major result (Theorem 11.1) in Ref. [8]. However, this assumption is not required for the derivation of Tsirelson's bound. It has thus been seen that the validity of Tsirelson's bound is not restricted to JBW-algebras and particularly not to Hilbert space quantum mechanics. The exceptional Jordan algebra (consisting of the self-adjoint 3×3 matrices the entries of which are octonions) is a concrete example of a JBW algebra which is not included in Hilbert space quantum mechanics, but still satisfies Tsirelson's bound.

If not all elements in A are observables, the sum of two observables is an element in A, but need not be an observable, and there is no linear structure for the observables anymore. However, is it so natural to postulate a linear structure for the observables? The observables in section 6 as well as those usually considered in quantum mechanics are real-valued observables. It is not very common to study complex-valued or vector-valued observables, but such a postulate would be equally natural for them, however, is not even valid in Hilbert space quantum mechanics. The usual real-valued observables are the self-adjoint operators ($a = a^*$) which form a real-linear structure; the complex-valued observables are the normal operators ($aa^* = a^*a$), but the sum of two normal operators is not normal unless they commute. One might therefore imagine a more general physical theory with no linear structure for the observables - not even for the usual real-valued ones.

Moreover, an interesting question is how the correlations would behave when the assumption that the squares of elements in the order-unit space A are positive is dropped or, perhaps more important, when third-order interference is permitted. It may not be expected that Tsirelson's bound remains valid. So the question is whether $c_{11} + c_{12} + c_{21} - c_{22}$ can then reach the value 4 (which is the algebraic maximum because $-1 \leq c_{11}, c_{12}, c_{21}, c_{22} \leq 1$), or whether there are other bounds between the two in these cases. An example where the algebraic maximum 4 is reached was found by Popescu and Rohrlich [10]; it can be implemented in a simple quantum logic, but it is not known whether this is possible in a quantum logic with a reasonable calculus of unique conditional probabilities.

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